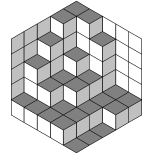




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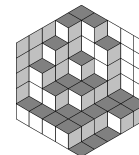


Quiz 1

- 1] Let  $P$  be an arbitrary point inside triangle  $ABC$ , denote by  $A_1$  (different from  $P$ ) the second intersection of line  $AP$  with the circumcircle of triangle  $PBC$ , define  $B_1, C_1$  similarly. Prove that  $(1 + 2 \cdot \frac{PA}{PA_1})(1 + 2 \cdot \frac{PB}{PB_1})(1 + 2 \cdot \frac{PC}{PC_1}) \geq 8$ .
- 2] Let  $n > 1$  be an integer, and  $n$  can divide  $2^{\phi(n)} + 3^{\phi(n)} + \dots + n^{\phi(n)}$ , let  $p_1, p_2, \dots, p_k$  be all distinct prime divisors of  $n$ . Show that  $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} + \frac{1}{p_1 p_2 \dots p_k}$  is an integer. (where  $\phi(n)$  is defined as the number of positive integers  $\leq n$  that are relatively prime to  $n$ .)
- 3] Determine the greatest positive integer  $n$  such that in three-dimensional space, there exist  $n$  points  $P_1, P_2, \dots, P_n$ , among  $n$  points no three points are collinear, and for arbitrary  $1 \leq i < j < k \leq n$ ,  $P_i P_j P_k$  isn't obtuse triangle.



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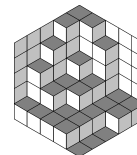


Quiz 2

- 1 Let  $ABC$  be a triangle, line  $l$  cuts its sides  $BC, CA, AB$  at  $D, E, F$ , respectively. Denote by  $O_1, O_2, O_3$  the circumcenters of triangle  $AEF, BFD, CDE$ , respectively. Prove that the orthocenter of triangle  $O_1O_2O_3$  lies on line  $l$ .
- 2 In a plane, there is an infinite triangular grid consists of equilateral triangles whose lengths of the sides are equal to 1, call the vertices of the triangles the lattice points, call two lattice points are adjacent if the distance between the two points is equal to 1; A jump game is played by two frogs  $A, B$ , "A jump" is called if the frogs jump from the point which it is lying on to its adjacent point, "A round jump of  $A, B$ " is called if first  $A$  jumps and then  $B$  by the following rules: Rule (1):  $A$  jumps once arbitrarily, then  $B$  jumps once in the same direction, or twice in the opposite direction; Rule (2): when  $A, B$  sits on adjacent lattice points, they carry out Rule (1) finishing a round jump, or  $A$  jumps twice continually, keep adjacent with  $B$  every time, and  $B$  rests on previous position; If the original positions of  $A, B$  are adjacent lattice points, determine whether for  $A$  and  $B$ , such that the one can exactly land on the original position of the other after a finite round jumps.
- 3 Let  $z_1, z_2, z_3$  be three complex numbers of which moduli are less than or equal to 1.  $w_1, w_2$  are two roots of the equation  $(z - z_1)(z - z_2) + (z - z_2)(z - z_3) + (z - z_3)(z - z_1) = 0$ . Prove that for  $j = 1, 2, 3$ ,  $\min\{|z_j - w_1|, |z_j - w_2|\} \leq 1$  holds.



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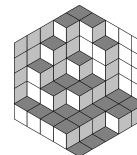


Quiz 3

- 1 Let  $P$  be the the isogonal conjugate of  $Q$  with respect to triangle  $ABC$ , and  $P, Q$  are in the interior of triangle  $ABC$ . Denote by  $O_1, O_2, O_3$  the circumcenters of triangle  $PBC, PCA, PAB$ ,  $O'_1, O'_2, O'_3$  the circumcenters of triangle  $QBC, QCA, QAB$ ,  $O$  the circumcenter of triangle  $O_1O_2O_3$ ,  $O'$  the circumcenter of triangle  $O'_1O'_2O'_3$ . Prove that  $OO'$  is parallel to  $PQ$ .
- 2 Prove that for arbitrary integer  $n > 16$ , there exists the set  $S$  that contains  $n$  positive integers and has the following property:if the subset  $A$  of  $S$  satisfies for arbitrary  $a, a' \in A, a \neq a', a + a' \notin S$  holds, then  $|A| \leq 4\sqrt{n}$ .
- 3 Let  $n > m > 1$  be odd integers, let  $f(x) = x^n + x^m + x + 1$ . Prove that  $f(x)$  can't be expressed as the product of two polynomials having integer coefficients and positive degrees.



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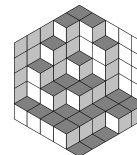


Quiz 4

- 1] Given a rectangle  $ABCD$ , let  $AB = b, AD = a (a \geq b)$ , three points  $X, Y, Z$  are put inside or on the boundary of the rectangle, arbitrarily. Find the maximum of the minimum of the distances between any two points among the three points. (Denote it by  $a, b$ )
- 2] Let  $x, y, z$  be positive real numbers, show that  $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} > 2\sqrt[3]{x^3 + y^3 + z^3}$ .
- 3] Let  $S$  be a set that contains  $n$  elements. Let  $A_1, A_2, \dots, A_k$  be  $k$  distinct subsets of  $S$ , where  $k \geq 2, |A_i| = a_i \geq 1 (1 \leq i \leq k)$ . Prove that the number of subsets of  $S$  that don't contain any  $A_i (1 \leq i \leq k)$  is greater than or equal to  $2^n \prod_{i=1}^k (1 - \frac{1}{2^{a_i}})$ .



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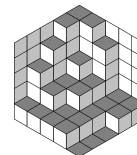


**Quiz 5**

- 1] Let  $ABC$  be an acute triangle, let  $M, N$  be the midpoints of minor arcs  $\widehat{CA}, \widehat{AB}$  of the circumcircle of triangle  $ABC$ , point  $D$  is the midpoint of segment  $MN$ , point  $G$  lies on minor arc  $\widehat{BC}$ . Denote by  $I, I_1, I_2$  the incenters of triangle  $ABC, ABG, ACG$  respectively. Let  $P$  be the second intersection of the circumcircle of triangle  $GI_1I_2$  with the circumcircle of triangle  $ABC$ . Prove that three points  $D, I, P$  are collinear.
- 2] For a given integer  $n \geq 2$ , determine the necessary and sufficient conditions that real numbers  $a_1, a_2, \dots, a_n$ , not all zero satisfy such that there exist integers  $0 < x_1 < x_2 < \dots < x_n$ , satisfying  $a_1x_1 + a_2x_2 + \dots + a_nx_n \geq 0$ .
- 3] Let  $0 < x_1 \leq \frac{x_2}{2} \leq \dots \leq \frac{x_n}{n}, 0 < y_n \leq y_{n-1} \leq \dots \leq y_1$ , Prove that  $(\sum_{k=1}^n x_k y_k)^2 \leq (\sum_{k=1}^n y_k)(\sum_{k=1}^n (x_k^2 - \frac{1}{4}x_k x_{k-1})y_k)$ . where  $x_0 = 0$ .



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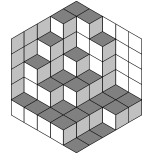


**Quiz 6**

- 1 Prove that in a plane, arbitrary  $n$  points can be overlapped by discs that the sum of all the diameters is less than  $n$ , and the distances between arbitrary two are greater than 1. (where the distances between two discs that have no common points are defined as that the distances between its centers subtract the sum of its radii; the distances between two discs that have common points are zero)
- 2 Prove that for all  $n \geq 2$ , there exists  $n$ -degree polynomial  $f(x) = x^n + a_1x^{n-1} + \dots + a_n$  such that (1)  $a_1, a_2, \dots, a_n$  all are unequal to 0; (2)  $f(x)$  can't be factorized into the product of two polynomials having integer coefficients and positive degrees; (3) for any integers  $x$ ,  $|f(x)|$  isn't prime numbers.
- 3 Find all positive integers  $n$  having the following properties: in two-dimensional Cartesian coordinates, there exists a convex  $n$  lattice polygon whose lengths of all sides are odd numbers, and unequal to each other. (where lattice polygon is defined as polygon whose coordinates of all vertices are integers in Cartesian coordinates.)



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**TST**

**Day 1**

- 1] Let  $ABC$  be a triangle, let  $AB > AC$ . Its incircle touches side  $BC$  at point  $E$ . Point  $D$  is the second intersection of the incircle with segment  $AE$  (different from  $E$ ). Point  $F$  (different from  $E$ ) is taken on segment  $AE$  such that  $CE = CF$ . The ray  $CF$  meets  $BD$  at point  $G$ . Show that  $CF = FG$ .
- 2] The sequence  $\{x_n\}$  is defined by  $x_1 = 2, x_2 = 12$ , and  $x_{n+2} = 6x_{n+1} - x_n, (n = 1, 2, \dots)$ . Let  $p$  be an odd prime number, let  $q$  be a prime divisor of  $x_p$ . Prove that if  $q \neq 2, 3$ , then  $q \geq 2p - 1$ .
- 3] Suppose that every positive integer has been given one of the colors red, blue, arbitrarily. Prove that there exists an infinite sequence of positive integers  $a_1 < a_2 < a_3 < \dots < a_n < \dots$ , such that infinite sequence of positive integers  $a_1, \frac{a_1 + a_2}{2}, a_2, \frac{a_2 + a_3}{2}, a_3, \frac{a_3 + a_4}{2}, \dots$  has the same color.

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**Day 2**

- 4] Prove that for arbitrary positive integer  $n \geq 4$ , there exists a permutation of the subsets that contain at least two elements of the set  $G_n = \{1, 2, 3, \dots, n\}$ :  $P_1, P_2, \dots, P_{2^n - n - 1}$  such that  $|P_i \cap P_{i+1}| = 2, i = 1, 2, \dots, 2^n - n - 2$ .
- 5] For two given positive integers  $m, n > 1$ , let  $a_{ij} (i = 1, 2, \dots, n, j = 1, 2, \dots, m)$  be nonnegative real numbers, not all zero, find the maximum and the minimum values of  $f$ , where 
$$f = \frac{n \sum_{i=1}^n (\sum_{j=1}^m a_{ij})^2 + m \sum_{j=1}^m (\sum_{i=1}^n a_{ij})^2}{(\sum_{i=1}^n \sum_{j=1}^m a_{ij})^2 + mn \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2}.$$
- 6] Find the maximal constant  $M$ , such that for arbitrary integer  $n \geq 3$ , there exist two sequences of positive real number  $a_1, a_2, \dots, a_n$ , and  $b_1, b_2, \dots, b_n$ , satisfying (1):  $\sum_{k=1}^n b_k = 1, 2b_k \geq b_{k-1} + b_{k+1}, k = 2, 3, \dots, n - 1$ ; (2):  $a_k^2 \leq 1 + \sum_{i=1}^k a_i b_i, k = 1, 2, 3, \dots, n, a_n \equiv M$ .