IMO Shortlist 2006

Algebra

$\boxed{1}$ $a_0,\ a_1,\ a_2,\ldots$ is a sequence of real numbers such that

$$a_{n+1} = [a_n] \cdot \{a_n\}$$

prove that exist j such that for every $i \geq j$ we have $a_{i+2} = a_i$.

$\boxed{2}$ Let a_0, a_1, a_2, ... be a sequence of reals such that $a_0 = -1$ and
$a_n + \dfrac{a_{n-1}}{2} + \dfrac{a_{n-2}}{3} + ... + \dfrac{a_1}{n} + \dfrac{a_0}{n+1} = 0$ for all $n \geq 1$.
Show that $a_n > 0$ for all $n \geq 1$.

$\boxed{3}$ The sequence $c_0, c_1, ..., c_n, ...$ is defined by $c_0 = 1, c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum\limits_{j \in J} c_j$, $y = \sum\limits_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β, and M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality $m < \alpha x + \beta y < M$ if and only if $(x, y) \in S$.

Remark: A sum over the elements of the empty set is assumed to be 0.

$\boxed{4}$ Prove the inequality:
$\sum\limits_{i<j} \dfrac{a_i a_j}{a_i + a_j} \leq \dfrac{n}{2(a_1 + a_2 + ... + a_n)} \cdot \sum\limits_{i<j} a_i a_j$
for positive reals a_1, a_2, ..., a_n.

$\boxed{5}$ If a, b, c are the sides of a triangle, prove that

$$\sum_{\text{cyc}} \frac{\sqrt{a+b-c}}{\sqrt{a} + \sqrt{b} - \sqrt{c}} \leq 3$$

$\boxed{6}$ Determine the least real number M such that the inequality $\Big| ab\left(a^2 - b^2\right) + bc\left(b^2 - c^2\right) + ca\left(c^2 - a^2\right) \Big| \leq M \big($
holds for all real numbers a, b and c.

Combinatorics

1 We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off. (b) Prove that there are infinitely many integers n for which the lamps will never be all off.

2 Let P be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

3 Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x:

$\sum_P x^{a(P)}(1-x)^{b(P)} = 1$, where the sum is taken over all convex polygons with vertices in S.

Alternative formulation:

Let M be a finite point set in the plane and no three points are collinear. A subset A of M will be called round if its elements is the set of vertices of a convex $A-$gon $V(A)$. For each round subset let $r(A)$ be the number of points from M which are exterior from the convex $A-$gon $V(A)$. Subsets with $0, 1$ and 2 elements are always round, its corresponding polygons are the empty set, a point or a segment, respectively (for which all other points that are not vertices of the polygon are exterior). For each round subset A of M construct the polynomial

$$P_A(x) = x^{|A|}(1-x)^{r(A)}.$$

Show that the sum of polynomials for all round subsets is exactly the polynomial $P(x) = 1$.

4 A cake has the form of an n x n square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement A. Let B be another such arrangement. Suppose that every grid rectangle with

one vertex at the top left corner of the cake contains no fewer strawberries of arrangement B than of arrangement A.

Prove that arrangement B can be obtained from A by performing a number of switches, defined as follows: A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

$\boxed{5}$ An $(n, k)-$ tournament is a contest with n players held in k rounds such that:

(i) Each player plays in each round, and every two players meet at most once. (ii) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.

Determine all pairs (n, k) for which there exists an $(n, k)-$ tournament.

$\boxed{6}$ A holey triangle is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

$\boxed{7}$ Consider a convex polyheadron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyheadron *antipodal* if there exist two parallel planes passing through these points and such that the polyheadron is contained between these planes. Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

Geometry

$\boxed{1}$ Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

$\boxed{2}$ Let ABC be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $\dfrac{AK}{KB} = \dfrac{DL}{LC}$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B and C are concylic.

$\boxed{3}$ Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE \quad , \quad \angle ABC = \angle ACD = \angle ADE$$

Let P be the point of intersection of the lines BD and CE. Prove that the line AP passes through the midpoint of the side CD.

$\boxed{4}$ Let ABC be a triangle such that $\widehat{ACB} < \widehat{BAC} < \dfrac{\pi}{2}$. Let D be a point of $[AC]$ such that $BD = BA$. The incircle of ABC touches $[AB]$ at K and $[AC]$ at L. Let J be the center of the incircle of BCD. Prove that (KL) intersects $[AJ]$ at its middle.

$\boxed{5}$ In triangle ABC, let J be the center of the excircle tangent to side BC at A_1 and to the extensions of the sides AC and AB at B_1 and C_1 respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

$\boxed{6}$ Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

$\boxed{7}$ In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices A, B, C, respectively. For $i \in \{a, b, c\}$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $\{i, j, k\} = \{a, b, c\}$) such that w_i lies on the opposite side of p_i than w_j and w_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

$\boxed{8}$ Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that $\angle PAB + \angle PDC \leq 90^\circ$ and $\angle PBA + \angle PCD \leq 90^\circ$. Prove that $AB + CD \geq BC + AD$.

$\boxed{9}$ Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Comment: This is my personal favourite of the ISL Geometry problems :D

$\boxed{10}$ Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Number Theory

$\boxed{1}$ Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

$\boxed{2}$ For $x \in (0,1)$ let $y \in (0,1)$ be the number whose n-th digit after the decimal point is the 2^n-th digit after the decimal point of x. Show that if x is rational then so is y.

$\boxed{3}$ We define a sequence $(a_1, a_2, a_3, ...)$ by setting

$$a_n = \frac{1}{n}\left(\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right]\right)$$

for every positive integer n. Hereby, for every real x, we denote by $[x]$ the integral part of x (this is the greatest integer which is $\leq x$).

a) Prove that there is an infinite number of positive integers n such that $a_{n+1} > a_n$. **b)** Prove that there is an infinite number of positive integers n such that $a_{n+1} < a_n$.

$\boxed{4}$ Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

$\boxed{5}$ Prove that the equation $\dfrac{x^7 - 1}{x - 1} = y^5 - 1$ doesn't have integer solutions!

$\boxed{6}$ Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer c, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that $ax + by = c$. An integer c is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$. Find all local champions and determine their number.

$\boxed{7}$ For all positive integers n, show that there exists a positive integer m such that n divides $2^m + m$.