

Algebra

- 1 a_0, a_1, a_2, \dots is a sequence of real numbers such that

$$a_{n+1} = [a_n] \cdot \{a_n\}$$

prove that exist j such that for every $i \geq j$ we have $a_{i+2} = a_i$.

- 2 Let a_0, a_1, a_2, \dots be a sequence of reals such that $a_0 = -1$ and

$$a_n + \frac{a_{n-1}}{2} + \frac{a_{n-2}}{3} + \dots + \frac{a_1}{n} + \frac{a_0}{n+1} = 0 \text{ for all } n \geq 1.$$

Show that $a_n > 0$ for all $n \geq 1$.

- 3 The sequence $c_0, c_1, \dots, c_n, \dots$ is defined by $c_0 = 1, c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j \in J} c_j, y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β , and M with

the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality $m < \alpha x + \beta y < M$ if and only if $(x, y) \in S$.

Remark: A sum over the elements of the empty set is assumed to be 0.

- 4 Prove the inequality:

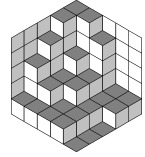
$$\sum_{i < j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \dots + a_n)} \cdot \sum_{i < j} a_i a_j$$

for positive reals a_1, a_2, \dots, a_n .

- 5 If a, b, c are the sides of a triangle, prove that

$$\sum_{\text{cyc}} \frac{\sqrt{a+b-c}}{\sqrt{a} + \sqrt{b} - \sqrt{c}} \leq 3$$

- 6 Determine the least real number M such that the inequality $|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leq M$ holds for all real numbers a, b and c .



Combinatorics

- 1 We have $n \geq 2$ lamps L_1, \dots, L_n in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off. (b) Prove that there are infinitely many integers n for which the lamps will never be all off.

- 2 Let P be a regular 2006-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P . The sides of P are also called *good*. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P . Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

- 3 Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S , let $a(P)$ be the number of vertices of P , and let $b(P)$ be the number of points of S which are outside P . A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x :

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1, \text{ where the sum is taken over all convex polygons with vertices in } S.$$

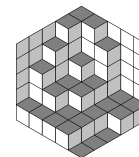
Alternative formulation:

Let M be a finite point set in the plane and no three points are collinear. A subset A of M will be called *round* if its elements is the set of vertices of a convex A -gon $V(A)$. For each round subset let $r(A)$ be the number of points from M which are exterior from the convex A -gon $V(A)$. Subsets with 0, 1 and 2 elements are always round, its corresponding polygons are the empty set, a point or a segment, respectively (for which all other points that are not vertices of the polygon are exterior). For each round subset A of M construct the polynomial

$$P_A(x) = x^{|A|}(1-x)^{r(A)}.$$

Show that the sum of polynomials for all round subsets is exactly the polynomial $P(x) = 1$.

- 4 A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement A . Let B be another such arrangement. Suppose that every grid rectangle with



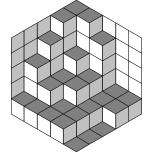
one vertex at the top left corner of the cake contains no fewer strawberries of arrangement B than of arrangement A .

Prove that arrangement B can be obtained from A by performing a number of switches, defined as follows: A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

- 5 An (n, k) - tournament is a contest with n players held in k rounds such that:
- (i) Each player plays in each round, and every two players meet at most once. (ii) If player A meets player B in round i , player C meets player D in round i , and player A meets player C in round j , then player B meets player D in round j .

Determine all pairs (n, k) for which there exists an (n, k) - tournament.

- 6 A holey triangle is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.
- 7 Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes. Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.



Geometry

- 1] Let ABC be triangle with incenter I . A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

- 2] Let ABC be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD , respectively, so that $\frac{AK}{KB} = \frac{DL}{LC}$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B and C are concyclic.

- 3] Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE \quad , \quad \angle ABC = \angle ACD = \angle ADE$$

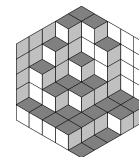
Let P be the point of intersection of the lines BD and CE . Prove that the line AP passes through the midpoint of the side CD .

- 4] Let ABC be a triangle such that $\widehat{ACB} < \widehat{BAC} < \frac{\pi}{2}$. Let D be a point of $[AC]$ such that $BD = BA$. The incircle of ABC touches $[AB]$ at K and $[AC]$ at L . Let J be the center of the incircle of BCD . Prove that (KL) intersects $[AJ]$ at its middle.

- 5] In triangle ABC , let J be the center of the excircle tangent to side BC at A_1 and to the extensions of the sides AC and AB at B_1 and C_1 respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D . Let E be the foot of the perpendicular from C_1 to line DJ . Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

- 6] Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D . Let AB be the diameter of w perpendicular to t , so that A, E, O_1 are on the same side of t . Prove that lines AO_1, BO_2, EF and t are concurrent.

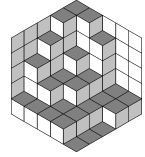
- 7] In a triangle ABC , let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB , respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC , not containing the vertices A, B, C , respectively. For $i \in \{a, b, c\}$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $\{i, j, k\} = \{a, b, c\}$) such that w_i lies on the opposite side of p_i than w_j and w_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.



- 8] Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that $\angle PAB + \angle PDC \leq 90^\circ$ and $\angle PBA + \angle PCD \leq 90^\circ$. Prove that $AB + CD \geq BC + AD$.
- 9] Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles $AB_1C_1, BC_1A_1, CA_1B_1$ intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Comment: This is my personal favourite of the ISL Geometry problems :D

- 10] Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P . Show that the sum of the areas assigned to the sides of P is at least twice the area of P .



Number Theory

- 1 Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

- 2 For $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose n -th digit after the decimal point is the 2^n -th digit after the decimal point of x . Show that if x is rational then so is y .

- 3 We define a sequence (a_1, a_2, a_3, \dots) by setting

$$a_n = \frac{1}{n} \left(\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} \right)$$

for every positive integer n . Hereby, for every real x , we denote by $[x]$ the integral part of x (this is the greatest integer which is $\leq x$).

a) Prove that there is an infinite number of positive integers n such that $a_{n+1} > a_n$. **b)** Prove that there is an infinite number of positive integers n such that $a_{n+1} < a_n$.

- 4 Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\dots P(P(x))\dots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

- 5 Prove that the equation $\frac{x^7 - 1}{x - 1} = y^5 - 1$ doesn't have integer solutions!

- 6 Let $a > b > 1$ be relatively prime positive integers. Define the weight of an integer c , denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that $ax + by = c$. An integer c is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$. Find all local champions and determine their number.

- 7 For all positive integers n , show that there exists a positive integer m such that n divides $2^m + m$.