Iran
National Math Olympiad (3rd Round)
2008

Algebra

1. Suppose that $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial. It is known that f has a root of norm larger than $\frac{3}{2}$. Prove that if α is a root of f then $f(\alpha^3+1) \neq 0$.

2. Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:

$$x\sqrt{y} + y\sqrt{z} + z\sqrt{x} \leq K\sqrt{(x+y)(y+z)(z+x)}$$

3. Let (b_0, b_1, b_2, b_3) be a permutation of the set $\{54, 72, 36, 108\}$. Prove that $x^5 + b_3x^3 + b_2x^2 + b_1x + b_0$ is irreducible in $\mathbb{Z}[x]$.

4. Let $x, y, z \in \mathbb{R}^+$ and $x + y + z = 3$. Prove that:

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27}(xy + xz + yz)$$

5. Prove that the following polynomial is irreducible in $\mathbb{Z}[x, y]$:

$$x^{200}y^5 + x^{51}y^{100} + x^{106} - 4x^{100}y^5 + x^{100} - 2y^{100} - 2x^6 + 4y^5 - 2$$

Combinatorics

1. Prove that the number of permutations α of $\{1,2,\dots,n\}$ and a subsets S of $\{1,2,\dots,n\}$ such that
$$\forall x \in S : \alpha(x) \notin S$$
is equal to $n!F_{n+1}$ in which F_n is the Fibonacci sequence such that $F_1 = F_2 = 1$

2. Prove that the number permutations α of $\{1,2,\dots,n\}$ s.t. there does not exist $i < j < n$ s.t. $\alpha(i) < \alpha(j+1) < \alpha(j)$ is equal to the number of partitions of that set.

3. Prove that for each n:
$$\sum_{k=1}^{n} \binom{n+k-1}{2k-1} = F_{2n}$$

4. Let S be a sequence that:
$$\begin{cases} S_0 = 0 \\ S_1 = 1 \\ S_n = S_{n-1} + S_{n-2} + F_n \quad (n > 1) \end{cases}$$
such that F_n is Fibonacci sequence such that $F_1 = F_2 = 1$. Find S_n in terms of Fibonacci numbers.

5. n people decide to play a game. There are $n-1$ ropes and each of its two ends are in hand of one of the players, in such a way that ropes and players form a tree. (Each person can hold more than rope end.)

   At each step a player gives one of the rope ends he is holding to another player. The goal is to make a path of length $n-1$ at the end.

   But the game regulations change before game starts. Everybody has to give one of his rope ends only two one of his neighbors. Let a and b be minimum steps for reaching to goal in these two games. Prove that $a = b$ if and only if by removing all players with one rope end (leaves of the tree) the remaining people are on a path. (the remaining graph is a path.) [img]http://i37.tinypic.com/2l9h1tv.png[/img]

Complex numbers

1. Prove that for $n > 0$ and $a \neq 0$ the polynomial $p(z) = az^{2n+1} + bz^{2n} + \bar{b}z + \bar{a}$ has a root on unit circle

2. Let $g, f : \mathbb{C} \longrightarrow \mathbb{C}$ be two continuous functions such that for each $z \neq 0$, $g(z) = f(\frac{1}{z})$. Prove that there is a $z \in \mathbb{C}$ such that $f(\frac{1}{z}) = f(-\bar{z})$

3. For each $c \in \mathbb{C}$, let $f_c(z, 0) = z$, and $f_c(z, n) = f_c(z, n-1)^2 + c$ for $n \geq 1$. a) Prove that if $|c| \leq \frac{1}{4}$ then there is a neighborhood U of origin such that for each $z \in U$ the sequence $f_c(z, n), n \in \mathbb{N}$ is bounded. b) Prove that if $c > \frac{1}{4}$ is a real number there is a neighborhood U of origin such that for each $z \in U$ the sequence $f_c(z, n), n \in \mathbb{N}$ is unbounded.

Final Exam

1. Police want to arrest on the famous criminals of the country whose name is Kaiser. Kaiser is in one of the streets of a square shaped city with n vertical streets and n horizontal streets. In the following cases how many police officers are needed to arrest Kaiser? [img]http://i38.tinypic.com/2i1icec$_th.png[/img][img]http : //i34.tinypic.com/28rk4s3_th.png[/img]a)Eachp$
$//i38.tinypic.com/35n615y.png[/img]$

3 a) Prove that there are two polynomials in $\mathbb{Z}[x]$ with at least one coefficient larger than 1387 such that coefficients of their product is in the set $\{-1, 0, 1\}$. b) Does there exist a multiple of x^2-3x+1 such that all of its coefficient are in the set $\{-1, 0, 1\}$

4. =A subset S of $\mathbb{R}^2$ is called an algebraic set if and only if there is a polynomial $p(x, y) \in \mathbb{R}[x, y]$ such that

$$S = \{(x, y) \in \mathbb{R}^2 | p(x, y) = 0\}$$

Are the following subsets of plane an algebraic sets? 1. A square [img]http://i36.tinypic.com/28uiaep.png[/img] 2. A closed half-circle [img]http://i37.tinypic.com/155m155.png[/img]

5. a) Suppose that $RBR'B'$ is a convex quadrilateral such that vertices R and R' have red color and vertices B and B' have blue color. We put k arbitrary points of colors blue and red in the quadrilateral such that no four of these $k+4$ point (except probably $RBR'B'$) lie one a circle. Prove that exactly one of the following cases occur? 1. There is a path from R to R' such that distance of every point on this path from one of red points is less than its distance from all blue points. 2. There is a path from B to B' such that distance of every point on this path from one of blue points is less than its distance from all red points. We call these two paths the blue path and the red path respectively.

Let n be a natural number. Two people play the following game. At each step one player puts a point in quadrilateral satisfying the above conditions. First player only puts red point and second player only puts blue points. Game finishes when every player has put n points on the plane. First player's goal is to make a red path from R to R' and the second player's goal is to make a blue path from B to B'. b) Prove that if $RBR'B'$ is rectangle then for each n the second player wins. c) Try to specify the winner for other quadrilaterals.

6. There are five research labs on Mars. Is it always possible to divide Mars to five connected congruent regions such that each region contains exactly on research lab. [img]http://i37.tinypic.com/f2iq8g.png[/img]

7. A graph is called a self-interesting graph if and only if it is isomorphic to a graph whose every edge is a segment and every two edges intersect. Notice that no edge contains a vertex except its two endings. a) Find all n's for which the cycle of length n is self-intersecting. b)

Prove that in a self-intersecting graph $|E(G)| \leq |V(G)|$. c) Find all self-intersecting graphs. [img]http://i35.tinypic.com/x43s5u.png[/img]

8 In an old script found in ruins of Perspolis is written: [code:1] This script has been finished in a year whose 13th power is 258145266804692077858261512663 You should know that if you are skilled in Arithmetics you will know the year this script is finished easily.[/code:1] Find the year the script is finished. Give a reason for your answer.

Geometry

1 Let ABC be a triangle with $BC > AC > AB$. Let A', B', C' be feet of perpendiculars from A, B, C to BC, AC, AB, such that $AA' = BB' = CC' = x$. Prove that: a) If $ABC \sim A'B'C'$ then $x = 2r$ b) Prove that if A', B' and C' are collinear, then $x = R + d$ or $x = R - d$.

(In this problem R is the radius of circumcircle, r is radius of incircle and $d = OI$)

2 Let l_a, l_b, l_c be three parallel lines passing through A, B, C respectively. Let l'_a be reflection of l_a into BC. l'_b and l'_c are defined similarly. Prove that l'_a, l'_b, l'_c are concurrent if and only if l_a is parallel to Euler line of triangle ABC.

3 Let $ABCD$ be a quadrilateral, and E be intersection points of AB, CD and AD, BC respectively. External bisectors of DAB and DCB intersect at P, external bisectors of ABC and ADC intersect at Q and external bisectors of AED and AFB intersect at R. Prove that P, Q, R are collinear.

4 Let ABC be an isosceles triangle with $AB = AC$, and D be midpoint of BC, and E be foot of altitude from C. Let H be orthocenter of ABC and N be midpoint of CE. AN intersects with circumcircle of triangle ABC at K. The tangent from C to circumcircle of ABC intersects with AD at F. Suppose that radical axis of circumcircles of CHA and CKF is BC. Find $\angle BAC$.

5 Let D, E, F be tangency point of incircle of triangle ABC with sides BC, AC, AB. DE and DF intersect the line from A parallel to BC at K and L. Prove that the Euler line of triangle DKL passes through Feuerbach point of triangle ABC.

Number Theory

1 Let $k > 1$ be an integer. Prove that there exists infinitely many natural numbers such as n such that:

$$n|1^n + 2^n + \cdots + k^n$$

2 Prove that there exists infinitely many primes p such that:

$$13|p^3 + 1$$

3 Let P be a regular polygon. A regular sub-polygon of P is a subset of vertices of P with at least two vertices such that divides the circumcircle to equal arcs. Prove that there is a subset of vertices of P such that its intersection with each regular sub-polygon has even number of vertices.

4 Let u be an odd number. Prove that $\dfrac{3^{3u} - 1}{3^u - 1}$ can be written as sum of two squares.

5 Find all polynomials $f \in \mathbb{Z}[x]$ such that for each $a, b, x \in \mathbb{N}$

$$a + b + c|f(a) + f(b) + f(c)$$