Grade 9

1. Do there exist 14 positive integers, upon increasing each of them by 1,their product increases exactly 2008 times?

2. Numbers a, b, c are such that the equation $x^3 + ax^2 + bx + c$ has three real roots.Prove that if $-2 \leq a + b + c \leq 0$,then at least one of these roots belongs to the segment $[0, 2]$

3. In a scalene triangle ABC, H and M are the orthocenter an centroid respectively. Consider the triangle formed by the lines through A, B and C perpendicular to AM, BM and CM respectively. Prove that the centroid of this triangle lies on the line MH.

4. There are several scientists collaborating in Niichavo. During an 8-hour working day, the scientists went to cafeteria, possibly several times.It is known that for every two scientist, the total time in which exactly one of them was in cafeteria is at least x hours ($x > 4$). What is the largest possible number of scientist that could work in Niichavo that day,in terms of x?

5. The distance between two cells of an infinite chessboard is defined as the minimum nuber to moves needed for a king for move from one to the other.One the board are chosen three cells on a pairwise distances equal to 100. How many cells are there that are on the distance 50 from each of the three cells?

6. The incircle of a triangle ABC touches the side AB and AC at respectively at X and Y. Let K be the midpoint of the arc AB on the circumcircle of ABC. Assume that XY bisects the segment AK. What are the possible measures of angle BAC?

7. A natural number is written on the blackboard. Whenever number x is written, one can write any of the numbers $2x + 1$ and $\frac{x}{x+2}$. At some moment the number 2008 appears on the blackboard. Show that it was there from the very beginning.

8. We are given 3^{2k} apparently identical coins,one of which is fake,being lighter than the others. We also dispose of three apparently identical balances without weights, one of which is broken (and yields outcomes unrelated to the actual situations). How can we find the fake coin in $3k + 1$ weighings?

Grade 10

1. Do there exist 14 positive integers, upon increasing each of them by 1,their product increases exactly 2008 times?

2. The columns of an $n \times n$ board are labeled 1 to n. The numbers $1, 2, ..., n$ are arranged in the board so that the numbers in each row and column are pairwise different. We call a cell "good" if the number in it is greater than the label of its column. For which n is there an arrangement in which each row contains equally many good cells?

3. A circle ω with center O is tangent to the rays of an angle BAC at B and C. Point Q is taken inside the angle BAC. Assume that point P on the segment AQ is such that $AQ \perp OP$. The line OP intersects the circumcircles ω_1 and ω_2 of triangles BPQ and CPQ again at points M and N. Prove that $OM = ON$.

4. The sequences $(a_n), (b_n)$ are defined by $a_1 = 1, b_1 = 2$ and
$a_{n+1} = \dfrac{1 + a_n + a_n b_n}{b_n}, b_{n+1} = \dfrac{1 + b_n + a_n b_n}{a_n}$.
Show that $a_{2008} < 5$.

5. Determine all triplets of real numbers x, y, z satisying: $1 + x^4 \leq 2(y - z)^2$, $1 + y^4 \leq 2(x - z)^2$, $1 + z^4 \leq 2(x - y)^2$.

6. In a scalene triangle ABC the altitudes AA_1 and CC_1 intersect at H, O is the circumcenter, and B_0 the midpoint of side AC. The line BO intersects side AC at P, while the lines BH and A_1C_1 meet at Q. Prove that the lines HB_0 and PQ are parallel.

7. For which integers $n > 1$ do there exist natural numbers $b_1, b_2, ..., b_n$ not all equal such that the number $(b_1 + k)(b_2 + k)...(b_n + k)$ is a power of an integer for each natural number k? (The exponenets may depend on k, but must be greater than 1)

8. On the cartesian plane are drawn several rectangles with the sides parallel to the coordinate axes. Assume that any two rectangles can be cut by a vertical or a horizontal line. Show that it's possible to draw one horizontal and one vertical line such that each rectangle is cut by at least one of these two lines.

Grade 11

1 Numbers a, b, c are such that the equation $x^3 + ax^2 + bx + c$ has three real roots.Prove that if $-2 \leq a + b + c \leq 0$,then at least one of these roots belongs to the segment $[0, 2]$

2 Petya and Vasya are given equal sets of N weights, in which the masses of any two weights are in ratio at most 1.25. Petya succeeded to divide his set into 10 groups of equal masses, while Vasya succeeded to divide his set into 11 groups of equal masses. Find the smallest possible N.

3 Given a finite set P of prime numbers, prove that there exists a positive integer x such that it can be written in the form $a^p + b^p$ (a, b are positive integers), for each $p \in P$, and cannot be written in that form for each p not in P.

4 Each face of a tetrahedron can be placed in a circle of radius 1. Show that the tetrahedron can be placed in a sphere of radius $\frac{3}{2\sqrt{2}}$.

5 The numbers from 51 to 150 are arranged in a 10×10 array. Can this be done in such a way that, for any two horizontally or vertically adjacent numbers a and b, at least one of the equations $x^2 - ax + b = 0$ and $x^2 - bx + a = 0$ has two integral roots?

6 A magician should determine the area of a hidden convex 2008-gon $A_1A_2 \cdots A_{2008}$. In each step he chooses two points on the perimeter, whereas the chosen points can be vertices or points dividing selected sides in selected ratios. Then his helper divides the polygon into two parts by the line through these two points and announces the area of the smaller of the two parts. Show that the magician can find the area of the polygon in 2006 steps.

7 In convex quadrilateral $ABCD$, the rays BA, CD meet at P, and the rays BC, AD meet at Q. H is the projection of D on PQ. Prove that there is a circle inscribed in $ABCD$ if and only if the incircles of triangles ADP, CDQ are visible from H under the same angle.

8 In a chess tournament $2n + 3$ players take part. Every two play exactly one match. The schedule is such that no two matches are played at the same time, and each player, after taking part in a match, is free in at least n next (consecutive) matches. Prove that one of the players who play in the opening match will also play in the closing match.